Active suspension with optimal control
based on a full vehicle model

ZHANG Jun-wei(张军伟), CHEN Si-zhong(陈思忠), ZHAO Yu-zhuang(赵玉壮)
(School of Mechanical Engineering, Beijing Institute of Technology, Beijing 100081, China)

Abstract: The 7-DOF model of a full vehicle with an active suspension is developed in this paper. The model is written into the state equation style. Actuator forces are treated as inputs in the state equations. Based on the basic optimal control theory, the optimal gains for the control system are figured out. So an optimal controller is developed and implemented using Matlab/Simulink, where the Riccati equation with coupling terms is deduced using the Hamilton equation. The all state feedback is chosen for the controller. The gains for all vehicle variables are traded off so that majority of indexes were up to optimal. The active suspension with optimal control is simulated in frequency domain and time domain separately, and compared with a passive suspension. Throughout all the simulation results, the optimal controller developed in this paper works well in the majority of instances. In all, the comfort and ride performance of the vehicle are improved under the active suspension with optimal control.

Key words: active suspension; full vehicle model; optimal control; frequency domain; time domain
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Suspension is a joint union between a body and tires. It plays the role of buffering the impact transferred from a terrain to the body and dissipating the vibration induced by the buffer. Finally, it reaches the purpose of keeping the vehicle riding smoothly. The main functions of a suspension are to minimize body vertical acceleration in order to increase passengers’ comfort, to minimize the wheel dynamic load for the purpose of providing good handling performance, and to react to dynamic deflection changes of the body that occur during maneuvering and load changes\(^1{–2}\). But it is almost impossible for a passive suspension to solve all the questions. We know that the passive suspension has a fixed spring stiffness and damping rate. It is only suitable for a limited terrain and a single frequency excitation. So a kind of suspension which provides a reaction force according to the movement of the body and tires is demanded to be developed. Because it has the capacity of adjusting itself to variable terrain actively, it is called active suspension.

The active suspension has been developed for many years. It was first proposed by Crossby and Karnopp in 1973. Since then, a lot of research has been done in structure design and control algorithm in past decades. Karnopp did a lot of work about the theoretical limitations, power requirements and isolation of active suspension\(^3{–5}\). Hedrick and Wormely did some research on active suspension design intensively\(^6\).

As for the control method, the skyhook damping control is an important concept in the first beginning\(^7\). Many innovative control methods have been brought into active suspension control based on the skyhook control\(^8{–10}\). Recently, more and more modern control methods have been introduced in active suspension control such as fuzzy logic control\(^11{–13}\), adaptive control\(^14{–15}\), \(H_{\infty}\) control\(^16\), optimal control\(^17{–19}\) and so
on. As for the optimal control method, the linear quadratic regulator (LQR) and linear quadratic Gaussian (LQG) have been used. In this paper, the LQR is chosen for active suspension control.

1 Full vehicle mathematical model

In order to consider all the four indexes, a full vehicle model is necessary to be built. A diagram of the full vehicle model with active suspension is represented in Fig. 1.

The full vehicle model has seven degrees of freedom (7-DOF) which contain three motions of a sprung mass (bounce, pitch and roll) and four motions of unsprung mass. This active suspension system has springs with constant stiffness and dampers with a constant damping coefficient between four wheels and the body. It is just like the passive suspension system. The main difference is that four actuators are set between the four wheels and the body. These four actuators are installed paralleled with the springs and dampers, which are the source of power for the whole active suspension system.

![Diagram of full vehicle model with 7-DOF](image)

**Fig. 1** Diagram of full vehicle model with 7-DOF

Before giving the mathematical model, some assumptions are necessary to be claimed as follows:

1. Mass separation coefficient is 1.
2. Pitch and roll angles are small enough.
3. Tire damping can be neglected.
4. Tire stiffness can be linear.

Based on these assumptions, the motion equations of sprung mass and unsprung mass are given in the following part.

As for the vertical motion of sprung mass, the motion equation can be described as

\[
m_s \ddot{z} = -(2K_{sr} + 2K_{sr}) z - (2C_{sf} + 2C_{sr}) \dot{z} +
\]

\[
(2aK_{sf} - 2bK_{sr}) \theta + (2aC_{sf} - 2bC_{sr}) \dot{\theta} +
\]

\[
K_{sf} \ddot{z}_{ufr} + K_{sf} \ddot{z}_{ufr} + K_{sr} \ddot{z}_{urr} + K_{sr} \ddot{z}_{urr} + C_{sf} \dot{z}_{ufr} + C_{sf} \dot{z}_{ufr} + C_{sr} \dot{z}_{urr} + C_{sr} \dot{z}_{urr} + F_{fr} + F_{fr} + F_{fr} + F_{fr} \] (1)

As for the pitch motion of sprung mass, the motion equation is described as

\[
I_{yy} \ddot{\theta} = (2aK_{sf} - 2bK_{sr}) z + (2aC_{sf} - 2bC_{sr}) \dot{z} +
\]

\[
(2a^2 K_{sf} + 2b^2 K_{sr}) \theta - aK_{sf} \dot{z}_{ufr} -
\]

\[
bK_{sf} \dot{z}_{ufr} + bK_{sf} \dot{z}_{ufr} + bK_{sr} \dot{z}_{urr} -
\]

\[
aC_{sf} \dot{z}_{ufr} - aC_{sf} \dot{z}_{ufr} + bC_{sr} \dot{z}_{ufr} +
\]

\[
bC_{sr} \dot{z}_{ufr} - aF_{fr} + bF_{fr} + bF_{fr} \] (2)

As for the roll motion of sprung mass, the motion equation can be described as

\[
I_{xx} \ddot{\varphi} = -\frac{1}{4} w^2 (2K_{sf} + 2K_{sr}) \varphi -
\]

\[
\frac{1}{4} w^2 (2C_{sf} + 2C_{sr}) \dot{\varphi} + \frac{w}{2} K_{sf} \dot{z}_{ufr} -
\]

\[
\frac{w}{2} K_{sf} \dot{z}_{ufr} + \frac{w}{2} K_{sf} \dot{z}_{ufr} - \frac{w}{2} K_{sr} \dot{z}_{ufr} + \frac{w}{2} C_{sf} \dot{z}_{ufr} -
\]
\[
\frac{w}{2} C_{sf} \ddot{z}_{ufl} + \frac{w}{2} C_{sr} \ddot{z}_{url} - \frac{w}{2} C_{sr} \dot{z}_{urr} + \frac{w}{2} F_{\lambda} - \\
\frac{w}{2} F_{fl} + \frac{w}{2} F_{fr} - \frac{w}{2} F_{rr}
\]

(3)

As for the vertical motion of four unsprung masses, the motion equations can be described as

\[m_u \ddot{z}_{ufl} = K_{sf} \ddot{z} + C_{sf} \dot{z} - aK_{sf} \theta - aC_{sf} \dot{\theta} + \]
\[\frac{w}{2} K_{sf} \phi + \frac{w}{2} C_{sf} \dot{\phi} - (K_{sf} + K_u) z_{ufl} - \\
C_{sr} \ddot{z}_{url} + K_u q_{fr} - F_{\lambda}
\]

(4)

\[m_u \ddot{z}_{ufr} = K_{sr} \ddot{z} + C_{sr} \dot{z} - aK_{sr} \theta - aC_{sr} \dot{\theta} - \]
\[\frac{w}{2} K_{sr} \phi - \frac{w}{2} C_{sr} \dot{\phi} - (K_{sr} + K_u) z_{ufr} - \\
C_{sr} \ddot{z}_{urr} + K_u q_{fr} - F_{fr}
\]

(5)

\[m_u \ddot{z}_{url} = K_{sr} \ddot{z} + C_{sr} \dot{z} + bK_{sr} \theta + bC_{sr} \dot{\theta} + \frac{w}{2} K_{sr} \phi + \\
\frac{w}{2} C_{sr} \dot{\phi} - (K_{sr} + K_u) z_{url} - C_{sr} \ddot{z}_{url} + K_u q_{rl} - F_{rl}
\]

(6)

\[m_u \ddot{z}_{urr} = K_{sr} \ddot{z} + C_{sr} \dot{z} + bK_{sr} \theta + bC_{sr} \dot{\theta} - \frac{w}{2} K_{sr} \phi - \\
\frac{w}{2} C_{sr} \dot{\phi} - (K_{sr} + K_u) z_{urr} - C_{sr} \ddot{z}_{urr} + K_u q_{rr} - F_{rr}
\]

(7)

where \( m_u \) is sprung mass; \( m_u \) = \( m_{ufl} = m_{ufr} = m_{url} = m_{urr} \) are four unsprung masses; \( K_{sf} \) = \( K_{sfl} = K_{sfr} \) are stiffness coefficients of front springs; \( K_{sr} \) = \( K_{srl} = K_{srr} \) are stiffness coefficients of rear springs; \( C_{sf} = C_{sfl} = C_{sfr} \) are damping coefficient of front dampers; \( C_{sr} = C_{srl} = C_{srr} \) are damping coefficients of rear dampers; \( K_u \) = \( K_{ufl} = K_{ufr} = K_{url} = K_{urr} \) are stiffness of tires; \( z \) is sprung mass position; \( \dot{z} \) is sprung mass velocity; \( \ddot{z} \) is sprung mass acceleration; \( z_{ufl}, z_{ufr}, z_{url}, z_{urr} \) are four unsprung masses’ positions; \( \dot{z}_{ufl}, \dot{z}_{ufr}, \dot{z}_{url}, \dot{z}_{urr} \) are four unsprung masses’ velocities; \( \ddot{z}_{ufl}, \ddot{z}_{ufr}, \ddot{z}_{url}, \ddot{z}_{urr} \) are four unsprung masses’ accelerations; \( q_{fl}, q_{fr}, q_{rl}, q_{rr} \) are road excitations to four tires; \( F_{fl}, F_{fr}, F_{rl}, F_{rr} \) are active damping forces induced by four dampers.

In order to build the model in Simulink®, the motion equations should be written in a state equations type. The state variables are defined as follows:

\[x_1 = z; x_2 = \dot{z}; x_3 = \theta; x_4 = \dot{\theta}; x_5 = \phi; x_6 = \dot{\phi};
\]
\[x_7 = \ddot{\phi}; x_8 = \ddot{z}_{ufl}; x_9 = \ddot{z}_{ufr}; x_{10} = \ddot{z}_{url}; x_{11} = \ddot{z}_{urr};
\] \[x_{12} = \ddot{z}_{ufr}; x_{13} = \ddot{z}_{urr}; x_{14} = \ddot{z}_{urr}.
\]

So the motion equations can be written into the state equations as follows:

\[
\dot{X} = AX + BU + LW
\]

\[Y = CX
\]

(8)

where

\[
\dot{X} = [\dot{x}_1 \ \dot{x}_2 \ \ldots \ \dot{x}_{13} \ \dot{x}_{14}]^T
\]

\[
X = [x_1 \ x_2 \ \ldots \ x_{13} \ x_{14}]^T
\]

\[
U = [F_{fl} \ F_{fr} \ F_{rl} \ F_{rr}]^T
\]

\[
W = [q_{fl} \ q_{fr} \ q_{rl} \ q_{rr}]^T
\]

Matrix \( A \) is a coefficient matrix for state variables, and matrix \( B \) is a coefficient matrix for input variables. They can be obtained by the motion equations. Matrix \( W \) is the road disturbances and \( U \) is the actuator forces which are treated as system inputs. They are the outer impact factors to the whole suspension system. The symbols \( F_{fl}, F_{fr}, F_{rl} \) and \( F_{rr} \) represent the forces generated by the four actuators which are our control objects. The symbols \( q_{fl}, q_{fr}, q_{rl} \) and \( q_{rr} \) represent the terrain disturbances on the four tires which are random. The output \( Y \) depends on the request of performance research.

2 Optimal control

2.1 Introduction to optimal control

Optimal control theory was developed in 1950s and has become the core of the modern control theory. Its main purpose is how to choose control signals so that the system performance reaches an optimal level under given conditions. Usually, there are multiple inputs and outputs for a control system. So the traditional control theory is not working in this case. Then the optimal control theory is brought into multiple inputs and outputs system.

Optimal control is separated into static optimal control and dynamic optimal control. Static optimal control means that system state reaches its optimum in a stable situation. All the parameters in the system do not change with time, but it reflects the static relationship when the system is in a stable situation. Majority of the production process can be controlled by static optimal con-
control and the control system surely gets a high accuracy. Dynamic optimal control means that the system state reaches its optimum when the system changes from one situation to another situation. All the parameters in the system change with time. Dynamic optimal control finds out an input matrix to make the characteristics indexes reach their optimum value under some constraint conditions. So the target function is not a normal function any longer, but a functional.

The optimal control characteristic index should be made in an optimal control system. The optimal index is chosen flexibly according to which system characteristic you want to study. Generally, the optimal index can be summarized as the following three types for a continuous system\(^{20}\):

1. **Synthetic index**
   Synthetic index is also called Bola index. It is described as
   \[
   J = \phi[X(t_f), t_f] + \int_{t_0}^{t_f} F[X(t), U(t), t] \, dt \quad (9)
   \]
   where \(\phi\) is the terminal index which is associated with the terminal time \(t_f\) and system state \(X(t_f)\), \(F\) is the dynamic index which is directly related to the system state vector \(X(t)\) and input \(U(t)\). This kind of characteristic index is called integrated index or Bola Index. It can be used to describe the minimum integral control under the terminal constraints.

2. **Integral index**
   Integral index is also called Lagrange index. It is a dynamic optimal control index. It can be described as
   \[
   J = \int_{t_0}^{t_f} F[X(t), U(t), t] \, dt \quad (10)
   \]
   It emphasizes the request that the system dynamic characteristic reaches its optimum value in the process of change.

3. **Terminal index**
   Terminal index is also called Meyer index. It is a totally terminal control index. It can be described as
   \[
   J = \phi[X(t_f), t_f] \quad (11)
   \]
   It only emphasizes the terminal state of the system reaching its optimum.

It is known that the active suspension control is a dynamic control question. It should be capable of changing constantly with the terrain. So the control index of active suspension is the second type.

2.2 **Optimal control for active suspension**

Active suspension control is a multi-objective control problem. It contains minimizing the body vertical acceleration to improve ride comfort, minimizing the pitch and roll angular acceleration to improve maneuverability, minimizing suspension’s dynamic deflection to improve passing ability, and minimizing the tire dynamic deflection to improve stability.

The target control system is a linear time varying system. Linear optimal control theory provides a systematic approach for the active suspension control.

To define the optimization problem, a cost index reflecting the engineering specifications must be set up. As signals change over time, the control objective should be oriented to minimize accumulated deviation. Large amplitudes of the manipulated actuators also imply the risk of saturation and energy consumption. Indeed, we hope the actuators consume less energy to maintain the system stability. So the optimal regulator must take into account penalties \(U(t)\) as well. In the settings, there will be different relative importance between the variables involved, so some form of relative weighting must be set up\(^{21}\). Finally, the optimal index can be written as

\[
J = \frac{1}{2} \int_{t_0}^{t_f} \left[ \sum_{i=1}^{m} k_i \ddot{y}_i^2(t) + \sum_{j=1}^{n} k_j \ddot{u}_j^2(t) \right] dt \quad (12)
\]

In view of the characteristics to be improved, the optimal index can be described as

\[
J = \frac{1}{2} \int_{t_0}^{t_f} \left[ k_1 \ddot{z}^2 + k_2 \dot{\theta}^2 + k_3 \ddot{\varphi}^2 + k_4 (z_{si} - z_{wi})^2 + k_5 (z_{wi} - q_j)^2 \right] dt \quad (13)
\]

where \(\ddot{z}\) is the body vertical acceleration, \(\dot{\theta}\) is the pitch angular acceleration, \(\ddot{\varphi}\) is the roll angular acceleration, \(z_{si} - z_{wi} (i = 4, 5, 6, 7)\) are the dy-
namic deflections of the four suspensions, and
\( z_{ij} - q_{ij} (j = 8, 9, 10, 11) \) are the deflections of the
two tires. The coefficients \( k_1 - k_{11} \) mean the perfor-

According to the above state Eqs. (1) – (7), the variables can be obtained as
\[
\begin{align*}
\ddot{z} &= E_1 X + F_1 U + G_1 W \\
\dot{\theta} &= E_2 X + F_2 U + G_2 W \\
\dot{\varphi} &= E_3 X + F_3 U + G_3 W \\
z_{si} - z_{wi} &= E_i X + F_i U + G_i W \\
z_{uj} - q_{ij} &= E_j X + F_j U + G_j W
\end{align*}
\]

where \( i = 4, 5, 6, 7; j = 8, 9, 10, 11 \).

Based on Eq. (14), the optimal index can be written as
\[
J = \frac{1}{2} \int_0^\infty \left[ X^T Q X + 2X^T N U + 2X^T M W + U^T R U + W^T S W \right] dt
\]
where \( Q, R \) and \( N, M \) and \( S \) are performance
weighting matrices which satisfy
\[
\begin{align*}
Q &= \sum_{t=1}^{11} E_i^T E_i \\
N &= \sum_{t=1}^{11} 2E_i^T F_t \\
M &= \sum_{t=1}^{11} 2E_i^T G_t, t = 1, 2, \cdots, 11 \\
R &= \sum_{t=1}^{11} F_t^T F_t \\
S &= \sum_{t=1}^{11} G_t^T G_t
\end{align*}
\]

If all system state variables are available and can be measured exactly, the control input that makes
the system performance index optimal is given in
\[
U = -KX
\]
where \( K \) is the optimal gain matrix which can be calculated by
\[
K = R^{-1}(B^T P + N^T)
\]
where \( B \) is given in Eq. (8), \( R \) and \( N \) are given
in Eq. (18), and \( P \) satisfies the Riccati
\[
PA + A^T P - (PB + N)R^{-1}(B^T P + N^T) + Q = 0
\]
The Riccati equation is deduced by the Hamilton function. \( P \) is calculated from Eq. (19), so the optimal input actuator forces can be obtained for the control system which are listed in
\[
\begin{align*}
F_{na} &= K_1 X \\
F_{ta} &= K_2 X \\
F_{ra} &= K_3 X \\
F_{rr} &= K_4 X
\end{align*}
\]
where \( K_1, K_2, K_3, K_4 \) are the optimal gains for the four actuator force.

The optimal control system flow chart is represented in Fig. 2.

So the closed loop optimal control state equation can be written as
\[
\dot{X} = [A - BR^{-1}(B^T P + N^T)]X + LW
\]
The closed loop optimal control chart is re-
3 Simulation and analysis

A kind of optimal controller is built in the previous part. We obtain an optimal coefficient matrix for the system input from the above optimal control calculation. After that we should use the computer to simulate it and check the control effect. The parameters for simulation come from a four wheels off-road vehicle. The parameters are listed in Tab. 1.

### 3.1 Road excitation

The road excitation is an important factor to properly verify the control effect. As for this research, I want to see the response of the suspension system to the road excitation, and compare the active suspension with the active suspension regarding the control effect to the road excitation. So the simple sine excitation signal is chosen here.

The sine signal is represented in Fig. 4.

<table>
<thead>
<tr>
<th>Tab. 1 Parameters for simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_s$/</td>
</tr>
<tr>
<td>kg</td>
</tr>
<tr>
<td>1 500</td>
</tr>
</tbody>
</table>

Fig. 4 Front and rear tire road excitation

The left chart in Fig. 4 is the road excitation for the front tire, and the right chart in Fig. 4 is the road excitation for the rear tire. The time interval is determined by the wheelbase and the velocity of the vehicle. These road excitations are added into the model as system inputs.

### 3.2 Simulation in frequency domain

The vehicle models equipped with the passive suspension and active suspension have been simulated under the sine road excitation. The power spectral density (PSD) of every vehicle variable is calculated using Welch method. The PSD of a selected set of vehicle variables are compared in this part. The variables related to vehicle behavior, such as the body vertical acceleration, pitch angular acceleration and roll angular acceleration, are shown in Fig. 5.

We are concerned with the scope of frequency from 1 Hz to 15 Hz. 1 Hz is the point of resonance of the vehicle body, and 10 Hz is the point of resonance of the tire. From Fig. 5a we can see that the active suspension works effectively in reducing the body vertical acceleration above the frequency of 5 Hz. In the frequency range from 0.1 Hz to 5 Hz, the control effect is not obvious. It means that the vehicle comfort is improved at higher frequency. In Fig. 5b, there is a similar situation. The pitch angular acceleration is decreased above the 6 Hz period. Although there is a little aggravation at low frequency, it does not impact keeping vehicle behavior at a stable level. In Fig. 5c, the control effect on the roll motion is similar to the pitch motion, while it has a little bit aggravation in the scope of low frequency.

The control effects on the ride performance, including the tire dynamic load and suspension deflection, are shown in Fig. 6.

From Fig. 6 we can get the information that
the active suspension does not work in the low frequency range from 0.1 Hz to 6 Hz even when there is a little aggravation. But it does play a role in decreasing the tire dynamic load and suspension deflection in the high frequency range around 10Hz. And you can see that the tire dynamic load and suspension deflection of vehicle equipped with active suspension reach steady state faster than the passive suspension.

In all, there is still a lot of potential for the control algorithm to be improved in the low frequency range. While the active suspension works well in the high frequency range and speeds up the attenuation of the relative motion between
sprung mass and unsprung mass.

3.3 Simulation in time domain

The power spectral density in frequency domain was discussed in the previous section. In this part, the characteristics in time domain are given. The vehicle variables related to behavior and ride are compared in time domain.

The simulation results related to behavior variables are shown in Fig. 7.

In Fig. 7a, the peak of the body vertical acceleration is reduced through the active suspension with optimal control. In other words, the comfort of the vehicle is improved. In Fig. 7b, the pitch angular acceleration does not get any improvement. In Fig. 7c, the roll angular acceleration is reduced which improves the stability when the vehicle is turning.

From Fig. 8 we can see that all the vehicle
variables related to the ride performance are improved. Both peaks of the tire dynamic load and suspension deflection are decreased. The curves of the active suspension tend to reach steady state faster than the passive suspension. It means that the active suspension can attenuate the motion of the tire effectively. In other words, the vehicle with the active suspension can obtain a superior ride performance.

Here the root mean square (RMS) of the body vertical acceleration, pitch angular acceleration and roll angular acceleration from 4 s to 6 s are given in Tab. 2.

<table>
<thead>
<tr>
<th>Suspension</th>
<th>Body vertical acceleration/(m·s⁻²)</th>
<th>Pitch angular acceleration/(rad·s⁻²)</th>
<th>Roll angular acceleration/(rad·s⁻²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>passive</td>
<td>1.0692</td>
<td>1.0598</td>
<td>0.7531×10⁻¹⁵</td>
</tr>
<tr>
<td>active</td>
<td>0.4880</td>
<td>0.8460</td>
<td>1.0803×10⁻¹⁵</td>
</tr>
</tbody>
</table>

From Tab. 2 we can get that the RMS of the body vertical acceleration and pitch angular acceleration have a large degree of reduction. The RMS of the roll angular acceleration increased a little. But the roll angular acceleration is pretty small, so it is of little impact on behavior of the vehicle. In all, the vehicle comfort is improved effectively.

4 Conclusions

In order to analyze the characteristics of the entire vehicle synthetically, it is necessary to build a full vehicle model. Based on this 7-DOF full vehicle model, the response of the vehicle to the road excitation can be checked through monitoring all the vehicle variables related to behavior and ride.

If the suspension system is treated as an independent unit, both the road excitation and actuator force should be treated as outside inputs for the suspension system. So, the road excitation and actuator force are brought into the suspension system as inputs. The characteristics of the entire vehicle result from the combined effects of the road excitation and actuators force.

In view of basic optimal control theory, a kind of optimal control method is used in the active suspension control system. An optimal controller is developed and the full state feedback is chosen for the controller. But it should be known that it is impossible to improve all the characteristics. There should be a trade-off according to the category and application of the vehicle.

Throughout the simulation results in frequency domain and time domain, the optimal controller developed in this paper works well. The majority of the vehicle variables are improved under the control of the optimal algorithm. In all, the vehicle equipped with the active suspension system has a better comfort and ride performance than the one with the passive suspension.

References:


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